BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Class field theory, by Jürgen Neukirch, Grundlehren der Mathematischen Wissenschaften, vol. 280, Springer-Verlag, Berlin, Heidelberg, New York, and Tokyo, 1986, $29.50. ISBN 0-387-15251-2

Local class field theory, by Kenkichi Iwasawa. Oxford Mathematical Monographs, Oxford University Press, New York and Clarendon Press, Oxford, 1986, viii + 155 pp., $39.95. ISBN 0-19-504030-9

Nowadays class field theory is mostly thought of as the theory which describes (more or less explicitly) the maximal abelian Galois extension of a local or global field K and which describes, again rather explicitly, the corresponding Galois group $\mathrm{Gal}(K^{ab}/K)$ and its quotients $\mathrm{Gal}(L/K)$, L/K abelian, preferably (and usually) in terms of some 'norm subgroups.' Here a local field is usually taken to be a finite algebraic extension of the field of p-adic integers $\mathbf{Q}_p$, or the field of Laurent series $\mathbf{F}_p((T))$ over a finite field, and a global field is a finite extension of the rationals $\mathbf{Q}$ or of the field of rational functions $\mathbf{F}_p(T)$ over a finite field $\mathbf{F}_p$.

It should be noted though, that there are more general local fields over which a class field theory can be developed, in particular complete, discretely valued fields with algebraically closed residue field [**9**], or, more generally, with perfect residue field [**3**]. And, much more importantly—in my opinion—, there is the algebraic K-theory based class field theory of Kato and Parshin [**6, 7, 8**] for finitely generated fields over their prime field (and schemes of finite type over $\mathbf{Z}$). However, these last named topics are not touched upon in the two books under review so I will not say much more about them.

The "definition" of class field theory given above is quite far removed from its origin—class field theory is one of those subjects which has gone through many "revolutions," generalizations, and changes of point of view; some 7 in my personal count and way of looking at it—and the description given does not give much of a clue to the origin of the word 'class field.'

Let us therefore backtrack a little bit, using [**1**] as a guide. An early concern is with polynomials $f(X)$ over the integers

$$f(X) = a_0X^m + a_1X^{m-1} + \cdots + a_m$$

$a_0a_m \neq 0$, $m \geq 1$. The prime p splits the polynomial (or $f(X)$ splits modulo p) if p does not divide a_0 and there are m different roots $x_1, x_2, \ldots, x_m$ to the equation (congruence) $f(X) \equiv 0 \bmod p$. A set of primes P splits a polynomial $f(X)$ if, with finitely many exceptions, all $p \in P$ split $f(X)$, and it splits a field K when the defining polynomial of some generator α of K over $\mathbf{Q}$ is split by P. A class field for a set of primes is then (if it exists) a maximal normal field over $\mathbf{Q}$ that is split by P such that, vice versa, P is maximal (with finitely many exceptions) for K. Consider for instance the quadratic forms

$$F_d(X, Y) = \begin{cases} X^2 - (d/4)Y^2 & \text{if } d \equiv 0 \mod 4, \\ X^2 - XY - \dfrac{d-1}{4}Y^2 & \text{if } d \equiv 1 \mod 4. \end{cases}$$

And consider the equations (to be solved in integers, X, Y, Z, p; p a prime; $(p, XY) = (Z, 2d) = 1$) and set (keeping the notation of [**1**]),

$$P_1 = \{p \colon Zp = F_d(X, Y) \text{ is solvable}\}$$
$$P_2 = \{p \colon Z^2p = F_d(X, Y) \text{ is solvable}\}$$
$$P_4 = \{p \colon p = F_d(X, Y) \text{ is solvable}\}$$

then the class fields of P_1, P_2, P_4 do exist and are known respectively as the splitting field of F_d (which is $\mathbf{Q}(\sqrt{d})$), the genus ring class field of F_d and the ring class field of F_d. The various quadratic reciprocity theorems (of Gauss and Jacobi) relate—among other things—also to such questions.

Subsequently, greatly stimulated by the Fermat problem, interest (also) focussed on the, clearly related, matter of unique factorization in (the ring of integers A_K of) a finite extension K of $\mathbf{Q}$. The ring A_K is defined as the ring of all $\alpha \in K$ which satisfy a monic equation $\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$ with $a_i \in \mathbf{Z}$. There is unique factorization (of ideals) in terms of prime ideals—this is where the term ideal comes from: it is short for ideal number—and thus the question arises how large an extension L of K to take such that all prime ideals of K become principal in L, i.e. such that $\mathfrak{p}A_L = (\beta)$ for a suitable $\beta \in A_L$, and related to this, the question arises how a prime ideal of K can split in an extension L, i.e. how $\mathfrak{p}A_L$ factorizes in terms of the prime ideals of L. In the case of the Fermat problem the relevant field K is the cyclotomic field $\mathbf{Q}(\xi_n)$ where ξ_n is a primitive nth root of unity: $\xi_n^n = 1$ and $\xi_n^d \neq 1$ for $d < n$. This led (Hilbert) to the notion of an (absolute) class field K_H for K. The ideal class group C_K of K is by definition the group of all ideals of K modulo the principal ideals of K. (A (fractional) ideal of K is a finitely generated A_K submodule of K; the submodules αA_K, $\alpha \in K$ are the principal (fractional) ideals.) The class number of K is the cardinality of C_K. The—it should be unique—class field K_H of K should be an abelian extension of K, with Galois group C_K such that K_H/K is unramified i.e. (neglecting infinite primes) if every prime ideal $\mathfrak{p}$ of K factorizes as $\mathfrak{q}_1 \cdots \mathfrak{q}_g$ for different prime ideals $\mathfrak{q}_1 \cdots \mathfrak{q}_g$ of K_H, and such that if f is the smallest integer such that $\mathfrak{p}^f$ is principal in K then $\mathfrak{p}$ decomposes as $\mathfrak{q}_1 \cdots \mathfrak{q}_g$ in K_H with $N_{K_H/H}(\mathfrak{q}_i) = \mathfrak{p}^f, fg = n$. Existence and uniqueness of such a class field was proved by Furtwängler. The Hilbert class field K_H has the property that every ideal $\mathfrak{a}$ of K becomes principal in K_H (the principal ideal theorem). Subsequently the idea of a class group was generalized to that of a ray class group (Weber, Takagi) and that made every finite abelian Galois extension a (generalized) class field and brings us back to the beginning of this preamble, namely the definition of class field theory as the construction of maximal abelian extensions of fields and the description of the corresponding (topological) Galois group.

Let us now first take up the subject of Iwasawa's beautiful book, the case of a local field with finite residue class field, i.e. the case of a finite extension K of $\mathbf{Q}_p$ or $\mathbf{F}_p((T))$. In this case the ring of integers A_K is a principal ideal ring with principal maximal ideal $\mathfrak{m}_k = (\pi_k)$ and group of units $U_K = A_K \backslash \mathfrak{m}_K$. There is a valuation ν_K on K defined by $\nu_K(x) = m$, if $x = \pi_K^m u$, $u \in U_K$, $\nu_K(0) = \infty$; the valuation defines a topology (and a norm): the $\mathfrak{m}_K^m$, $m = 1, 2, \ldots$ form a system of open neighborhoods of 0. It turns out that the Galois group of the maximal abelian extension is isomorphic to $U_K \times \hat{\mathbf{Z}}$, where $\hat{\mathbf{Z}} = \varprojlim \mathbf{Z}/(n)$ is the profinite completion of the integers. More precisely let L/K be a finite abelian extension, then there is a canonical (and functorial in a suitable sense) isomorphism

$$K^*/N_{L/K}L^* \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

where $K^* = K\backslash\{0\}$ and $N_{L/K} \colon L^* \to K^*$ is the norm map.

$$N_{L/K}(\beta) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\beta)$$

and these reciprocity isomorphisms combine to define an isomorphism

$$U_K \times \hat{\mathbf{Z}} = \hat{K}^* \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)$$

where $\hat{K}^*$ is the completion of the group K^* with respect to the topology of closed subgroups of finite index in K^* (where a closed subgroup is defined via the topology inherited from the topology defined by the valuation on K). Implicit in this statement is the fact that the subgroups of K^* which arise as a norm subgroup are precisely the closed subgroups of finite index (the existence theorem and traditionally one of the harder parts of a class field theory). The canonical maps $K^* \to \mathrm{Gal}(L/K)$ giving the isomorphism are known as reciprocity maps or norm residue symbols because of their relations with Gauss, Jacobi, Legendre, Hilbert, and Artin reciprocity and norm residue symbols.

In Iwasawa's elegant treatment of all this (and more) the rather explicit construction of the maximal abelian extension by Lubin and Tate is most important, as it is in several other approaches including the one in Neukirch's book. Possibly the fastest way to describe the maximal abelian extension is as follows. Choose a $\pi_K \in \mathfrak{m}_K$ such that $(\pi_K) = \mathfrak{m}_K$. Let q be the number of elements of the residue field $A_K/\mathfrak{m}_K = k$. Consider the polynomial

$$f(X) = \pi_K X + X^q.$$

Let $f^{(m)}(X)$ be the nth iterate of $f(X)$, i.e. $f^{(1)}(X) = f(X)$ and $f^{(m)}(X) = f(f^{(m-1)}(X))$. Now let W_m be the set of all roots of $f^{(m)}(X)$ (in some algebraic closure of K). Let L_m be the field $K(W_m)$. Then L_m/K is a totally ramified abelian extension with Galois group U_K/U_K^m where

$$U_K^m = \{u \in U_K : u \equiv 1 \bmod \pi_K^m\}.$$

Given the right point of view, which is that $f(X)$ is in fact an endomorphism of a formal group over A_K, these facts are rather easy to prove. Much harder is then to show that $N_{L/K} U_{L_m} = U_K^m$ which can be done in a variety of ways and it is here that—in my opinion—there are the most essential differences between the various treatments.

These formal groups, which are formal groups with maximally large endomorphism groups, are the analogues of elliptic curves with complex multiplication in the case of global class field theory of imaginary quadratic fields and thus the Lubin-Tate theory becomes very analogous to the Kronecker-Weber and Weber-Takagi theorems concerning abelian extensions of **Q** and of imaginary quadratic fields respectively. Together with a maximal unramified extension K_{nr} of K the L_m generate K^{ab}, i.e. $(\bigcup_m L_m)K_{nr} = K^{ab}$. Once one has sorted out the dependence on π of the L_m there also result, modulo a lot of additional work, 'explicit' formulas for the residue maps due to Wiles, Coleman, de Shalit, and others, which are important in a number of applications [**2**, **10**]. This is the topic of the last chapter of Iwasawa's book. It concludes with appendices on Galois cohomology; the Brauer group of a local field, essential to the cohomological approach to local class field theory, and an outline of part of Hazewinkel's approach [**4**], to local class field theory. The latter method was the basis of a previous treatment by Iwasawa of local class field theory [**5**] of which this book is a really completely rewritten and reworked version.

Now let us turn to the second book under review, by Jürgen Neukrich. He adds yet another layer of abstractness to the already towering tower of abstract layers of class field theory. That sounds like bad news. But the result is such an elegant unified presentation of both local and global class field theory (but excluding as far as is known—this is certainly something which merits investigation—the class field theories for generalized local fields and the algebraic K-theory based Parshin-Kato class field theory, I mentioned at the beginning), that I am greatly inclined to count this as one of my seven 'revolutions' in class field theory. The basic idea consists of a generalization: an abstract class field theory which is purely group theoretical. What is needed first is a profinite group G, i.e. a directed projective limit of finite groups (whence the name). Now use symbols K to label the closed subgroups of finite index of G. (The model to keep in mind is the (topological) Galois group of an infinite Galois extension of a field; in that case the label L of G_L is the fix-field of the closed subgroup of finite index G_L.) Let the group G itself have the label K. Given two labels M and L one says that M/L is a Galois extension if the group G_M is a normal subgroup of G_L. The Galois group is then $\mathrm{Gal}(M/L) = G_L/G_M$. Let the label of $\{1\} \subset G$ be $\overline{K}$ (the algebraic closure of K). The composite LM of two labels is the label belonging to the group (topologically) generated by G_L and G_M. One also sets $[M\colon L] = \#\mathrm{Gal}(M/L)$ if M/L is Galois. The two other things one needs for an abstract class field theory are now a surjective continuous homomorphism

$$\deg\colon G \to \hat{\mathbf{Z}}$$

and a suitable G-module$A$ (such that G acts continuously on A where A has the discrete topology). Let $\tilde{K}$ be the label of Ker(deg) and for each L/K let $\tilde{L} = \tilde{K}L$. Then $\mathrm{Gal}(\tilde{L}/L) \simeq \hat{\mathbf{Z}}$ and one defines $f_L = [L \cap \tilde{K}\colon K]$. Using the module A one defines

$$A_L = \{a \in A\colon \sigma(a) = a \text{ for all } \sigma \in G_L\}$$
$$N_{M/L}\colon A_M \to A_L, a \mapsto \prod_{\sigma\in\mathrm{Gal}(M/L)} \sigma(a)$$

(A is written multiplicatively). The extra structure one needs on A for an abstract class field theory is now what the author calls a Henselian valuation with respect to deg, which is a homomorphism

$$\nu\colon A_K \to \hat{\mathbf{Z}}$$

compatible with deg in the sense that

$$\nu(A_K) = Z \supset \mathbf{Z}, \qquad Z/nZ \simeq \mathbf{Z}/n\mathbf{Z}, \qquad \nu(N_{L/K}A_L) = f_L Z$$

and finally the G-module A is required to satisfy the low dimension cohomological conditions

$$\#H^0(\mathrm{Gal}(M/L), A_M) = \#\mathrm{Gal}(L/M)$$
$$\#H^{-1}(\mathrm{Gal}(M/L), A_M) = 1$$

(which last condition is the abstract version of Hilbert 90). This is all that is needed to define, fairly straightforwardly, abstract reciprocity maps

$$r_{M/L}\colon \mathrm{Gal}(M/L) \to A_M/N_{M/L}A_L.$$

To interpret all this it is easiest to keep the case of a characteristic zero local field (with finite residue field) in mind. In that case $\tilde{K}$ is the maximal unramified extension of K, f_L is the residue degree of L/K, $A = K^*$, etc. And indeed it takes the author in Chapter III some 7 pp. to verify that his axioms are satisfied in this case. (There are also manifold other more explicit results in Chapter III such as the local and global Kronecker-Weber theorems that abelian extensions of $\mathbf{Q}_p$ resp. $\mathbf{Q}$ are contained in cyclotomic extensions, the Hilbert symbol, and the Lubin-Tate construction already discussed above.

The global case is, as always nowadays, more difficult, though historically it came first, another of those revolutions in class field theory. In this case A is the idèle class group of the global field K. An idèle is simply a sequence $(\alpha_\mathfrak{p})$ where $\mathfrak{p}$ runs through all primes of K such that $\alpha_\mathfrak{p} \in K_\mathfrak{p}^*$, where $K_\mathfrak{p}$ is the completion of K with respect to the valuation defined by the ideal $\mathfrak{p}$, and such that $\alpha_\mathfrak{p}$ is a unit in $K_\mathfrak{p}$ for almost all $\mathfrak{p}$. (For an infinite prime $\mathfrak{p}$ the corresponding group of units is all of $K_{\mathfrak{p}}^*$, for a finite prime $\mathfrak{p}$ it is the group of elements $U_{\mathfrak{p}}$ of $\mathfrak{p}$-valuation zero in $K_{\mathfrak{p}}$.) Let I_K denote the group of idèles. The principal idèles are the elements of K^* (under the diagonal embedding $K^* \hookrightarrow \prod_{\mathfrak{p}} K_{\mathfrak{p}}^*$) and the quotient is the idèle class group $C_K = I_K/K^*$. Incidentally, as I learned from the book under review, the word idèle comes from ideal element, which got abbreviated id.el. whence (in French, the concept originated with Chevalley) idèle. Thus 'ideal' and 'idèle' have essentially same root. For global class field theory the G-module A is now the idèle class group C_K. There is, as expected, a great deal more to do to establish the author's axioms in this case, but still the treatment is basically painless. I will not give more details; this review is already getting out of hand. Then in §8 of Chapter IV the author proceeds to link the idèle theoretic formulation where the basic isomorphism is

$$\mathrm{Gal}(L/K) \overset{\simeq}{\longleftarrow} C_K/N_{L/K}C_L$$

with the older ideal theoretic formulation, i.e. ray class fields. The starting connection between idèles and idèle class group and ideals and ideal class group is

$$I_K/I_K^{\infty} \simeq J_K, \qquad I_K/I_K^{\infty}K^* \simeq J_K/P_K$$

where J_K and P_K are respectively the group of (fractional) ideals and (fractional) principal ideals and I_K^{∞} is the subgroup

$$I_K^{\infty} = \prod_{\mathfrak{p}\text{ infinite}} K_{\mathfrak{p}}^* \times \prod_{\mathfrak{p}\text{ finite}} U_{\mathfrak{p}}.$$

And thus the author recovers all the theorems which are sometimes listed as the principal theorems of (ray) class field theory: abelian extensions are ray class fields; for any ray class group H there exists a unique corresponding abelian extension; the decomposition of primes theorem; the conductor-ramification theorem; the translation theorem; Artin's general reciprocity law, the principal ideal theorem; not that these theorems all occur under these (or related) names in the index. And that brings me to perhaps the only thing to cavil about in this admirable book. For a book about a topic so laden with named theorems and with quite generally an unusually heavy terminological load, the index is decidedly skimpy. The reader who wants to look up specific theorems, concepts, and definitions rather than read (i.e. work) through the book systematically will have a tough time of it. An index of symbols would also have helped. Though, come to think about it, even in this aspect, the weakest of the book, it is probably better than most; the art of indexing in any case seems to be a practically forgotten one.

There is more to class field theory, as I have hinted occasionally above, than is covered in these books, and there is much more in these books than I have had occasion to mention.

All in all, we have here two excellent books which are more than heartily recommended to departmental and individual collections alike, and which I am happy to add to my own. There is only one thing wrong with that, being the excellent books they are, I obtained copies before they were offered to me for review. Ah well!

REFERENCES

1. H. Cohn, *Introduction to the construction of class fields*, Cambridge Univ. Press, 1985.

2. E. De Shalit, *The explicit reciprocity law in local class field theory*, Duke Math. J. **53** (1986), 163–176.

3. M. Hazewinkel, *Classes de corps local*, Appendice dans M. Demazure, P. Gabriel, Groupes algébriques, tome 1, North-Holland, 1971.

4. ——, *Local class field theory is easy*, Adv. in Math. **18** (1975), 148–181.

5. K. Iwasawa, *Local class field theory*, Iwanami-Shoten, 1980 (In Japanese).

6. Kazuya Kato, *Class field theory and algebraic K-theory*, Algebraic geometry, Lecture Notes in Math., vol. **1016**, Springer-Verlag, Berlin and New York, 1983, pp. 109–126.

7. ——, *Vanishing cycles, ramification of valuations, and class field theory*, Duke Math. J. **55** (1987), 629–661.

8. A. N. Parshin, *Local class field theory*, Proc. Steklov Inst. Math. **165** (1985), 157–185.

9. J.-P. Serre, *Sur les corps locaux à corps résiduely algébriquement clos*, Bull. Soc. Math. France **89** (1961), 105–154.

10. A. Wiles, *Higher explicit reciprocity laws*, Ann. of Math (2) **107** (1987), 235–254.

MICHIEL HAZEWINKEL
CWI, AMSTERDAM

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 1, July 1989

0273-0979/89 $1.00 + $.25 per page

Multiphase averaging for classical systems, with applications to adiabatic theorems by Pierre Lochak and Claude Meunier. (Translated by H. S. Dumas), Applied Mathematical Sciences, vol. 72, Springer-Verlag, New York, Berlin, Heidelberg, 1988, xi + 360 pp., $39.80. ISBN 0-387-96778-8

The idea of a separation of scales is of fundamental importance in our attempts to understand the world. When we speak of movement up or down "on the average," we are appealing to a process which removes rapid fluctuations and uncovers underlying trends. The formal perturbation procedure known as the method of multiple scales (or, in its simplest form, two-timing) relies on such a separation of time scales, as do the various averaging and homogenization theorems which make up an important part of the theory of differential equations and which form the subject of the book under review.

The simplest form of averaging, over a single time scale, proceeds as follows. Starting with a sufficiently smooth vector field $f(x, t)$ on $\mathbf{R}^n \times \mathbf{R}$ which depends T-periodically on time, t, the *averaged vector field* is defined as

$$\bar{f}(x) = \frac{1}{T}\int_0^T f(x, t)\,dt \qquad (0)$$